Sensitivity analysis for nonlinear hyperbolic equations

Laboratoire de mathématiques de Versailles - CNRS UMR 8100

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Sensitivity Analysis:

the study of how **variations in the output** of a model can be attributed to different sources of **uncertainty in the model input**.

Model: PDEs with i.c and b.c.
depending on parameters

parameters = model inputs
state variable = model output

Possible applications:

- optimisation
- estimate of close solutions
- uncertainty propagation

Continuous approach

"differentiate then discretise"

👍 well-suited for a well-defined reduced set of equations

👎 difficult to deal with changing domains

non consistent with discrete evaluation of perturbed configuration

Discrete approach

"discretise then differentiate"

👍 well-suited for realistic situations

👎 no theoretical insight

must be repeated for new situations

differentiation of all numerical tricks required

State equations

Let $u(x,t) \in \mathbb{R}^n$ be the vector of the state variables.

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$\begin{cases} \partial_t u + A(u)\partial_x u = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$A(u) = \frac{\partial f}{\partial u}$$

If *A(u)* is **diagonalisable**, the system is **strictly hyperbolic**.

Sensitivity equations

Let $\partial_p u = u_p$ be the sensitivity with respect to the parameter p.

By a formal differentiation of the state equations, one obtains:

$$\begin{cases} \partial_t u_p + \partial_x (A(u) u_p) = 0 & x \in \mathbb{R},\ t > 0 \\ u_p(x,0) = \partial_p g(x) & x \in \mathbb{R} \end{cases}$$

$$\begin{cases} \partial_t u_p + A(u)\partial_x u_p + \partial_p (A(u))\partial_x u = 0 & x \in \mathbb{R},\ t > 0 \\ u_p(x,0) = \partial_p g(x) & x \in \mathbb{R} \end{cases}$$

Global system

By defining

$$\mathbf{U} = \begin{bmatrix} u \\ u_p \end{bmatrix} \qquad M(\mathbf{U}) = \begin{bmatrix} A(u) & 0 \\ \partial_p(A(u)) & A(u) \end{bmatrix}$$

one can write:

$$\begin{cases} \partial_t \mathbf{U} + M(\mathbf{U})\partial_x \mathbf{U} = 0 & x \in \mathbb{R},\ t > 0 \\ \mathbf{U}(x,0) = \mathbf{G}(x) & x \in \mathbb{R} \end{cases}$$

Generally speaking, the global system is **weakly hyperbolic**.

Weakly hyperbolic systems

Features of weakly hyperbolic systems: generation of **Dirac's distribution**.

Example: Burger's equation

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ \partial_t u_p + u\partial_x(u_p) + \partial_x(u)u_p = 0 \end{cases}$$

If we consider the Riemann problem:

$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

with $u_L > u_R$, in the sensitivity

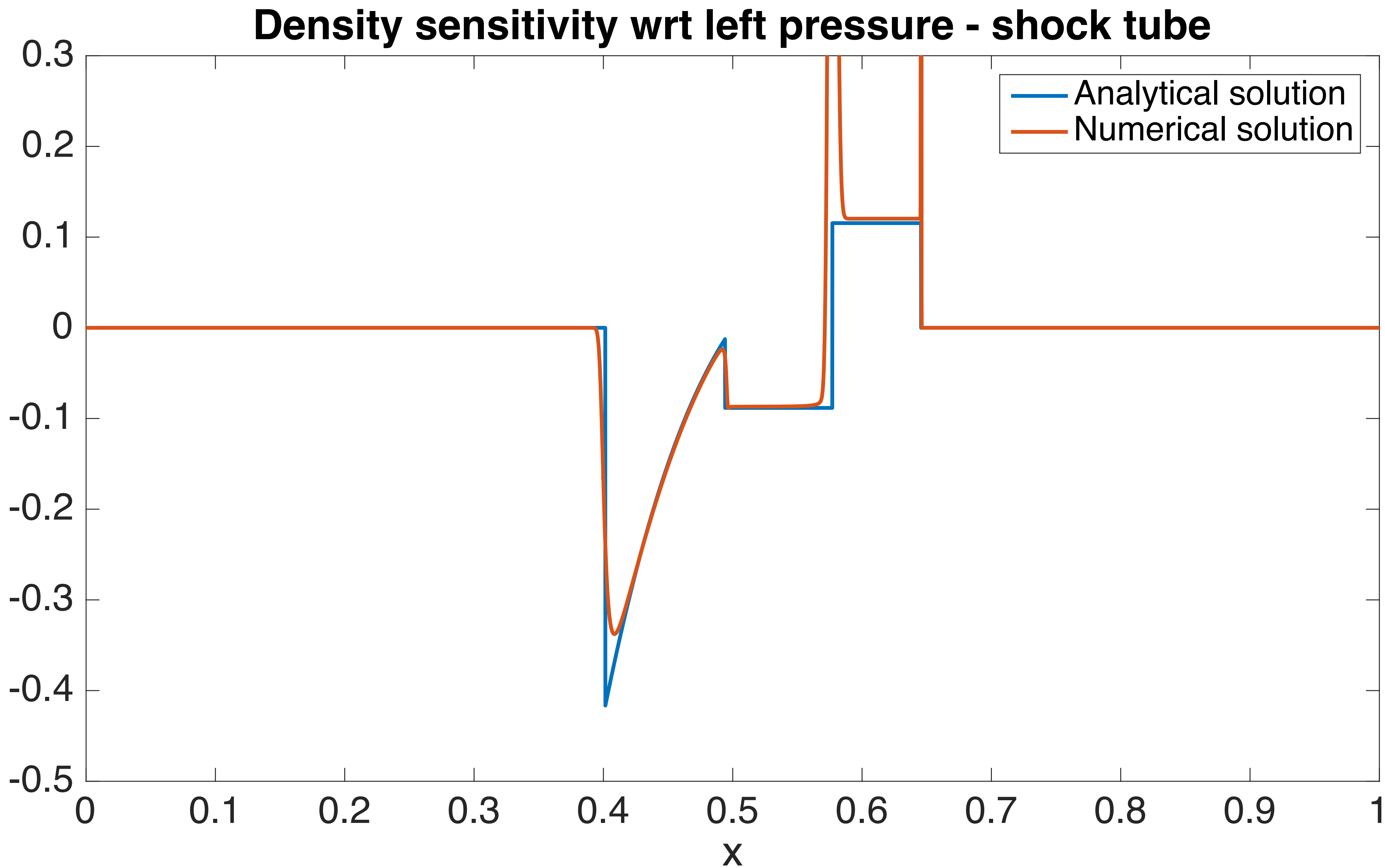
there's a source term with as a coefficient $\partial_x u$

which generates a Dirac's distribution.

From another point of view: the sensitivity is the **distributional derivative** of the state.

Weakly hyperbolic systems

$$\|u_p(x,t) - u_{p,i}^n\| \nrightarrow 0 \;\; h \to 0$$



AIM: remove the Dirac from the simulations

Numerical solution

Suggested strategy

Step 1: solution of the state with numerical methods for hyperbolic system

$$\partial_t u + A(u)\partial_x u = 0$$

Step 2: solution of the sensitivity with a corrected numerical scheme

$$\partial_t u_p + \partial_p(A(u))\partial_x u + A(u)\partial_x u_p = 0$$

Suggested strategy

Step 1: solution of the state with numerical methods for hyperbolic system

$$\partial_t u + A(u)\partial_x u = 0$$

Finite Volumes schemes:

$$u_j^{n+1} = u_j^n + \frac{\Delta x}{\Delta t}(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}})$$

HLL	Roe
 simple implementation	 solves exactly the shock
 neglect contact discontinuities	 non entropic

Suggested strategy

Step 2: solution of the sensitivity with a corrected numerical scheme

$$\partial_t u_p + \partial_p(A(u))\partial_x u + A(u)\partial_x u_p = 0$$

Idea: apply the same scheme used for the state plus a correction.

Rankine-Hugoniot conditions for the state:

$$f(u_L) - f(u_R) = (u_L - u_R)c_s$$

Differentiation with respect to the parameter:

$$\partial_p(f(u_L) - f(u_R)) = (u_{p,L} - u_{p,R})c_s + \boxed{(u_L - u_R)\partial_p c_s}$$

Difficulties:

- estimate of $\partial_p c_s$
- shock detection

Burger's equation

State equation:

$$\begin{cases} \partial_t u + u \partial_x u = 0 \\ u(x,0) = g(x) \end{cases}$$

Sensitivity equation:

$$\begin{cases} \partial_t u_p + u \partial_x u_p + u_p \partial_x u = 0 \\ u_p(x,0) = h(x) \end{cases}$$

Matrix of the global system:

$$M(\mathbf{U}) = \begin{bmatrix} u & 0 \\ u_a & u \end{bmatrix}$$

M is **not diagonalisable**

It is possible to find an **analytical solution** using the **method of characteristics**

We look for a curve $x_c(t)$ such that:

$$0 = \frac{d}{dt}u(x_c(t),t) = \frac{\partial}{\partial x}u(x_c(t),t)\frac{dx_c}{dt} + \frac{\partial}{\partial t}u(x_c(t),t),$$

$$\Rightarrow \frac{dx_c}{dt} = u(x_c(t),t),\ \ x_c(0) = x_0$$

Remark: $u(x_c(t),t) \equiv u(x_c(0),0) = g(x_0) \quad \forall t > 0$

The characteristics are **straight lines**

Burger's equation

Problem: intersection of characteristics

$$x_{c,1}(t) = x_1 + g(x_1)t \qquad x_{c,2}(t) = x_2 + g(x_2)t$$

Breaking time: the smallest *t* such that $x_{c,1}(t) = x_{c,2}(t)$

$$t_s := -\frac{1}{\min g'(x)}$$

- if $g'(x) > 0 \quad \forall x$ no intersection, the method is valid $\forall t$
- if $\exists x \ : \ g'(x) < 0$ the method is valid $\forall t < t_s$

The **state** solution is known in an **implicit** form:

$$u(x,t) = g(x_0 - u(x,t)t)$$

By differentiating it, we find an **explicit** form for the **sensitivity**:

$$u_p(x,t;p) = \frac{\partial}{\partial x} g(x - u(x,t;p)t;p)(-u_p(x,t;p)t) + \frac{\partial}{\partial p} g(x - u(x,t;p)t;p)$$

$$u_p(x,t) = \frac{h(x - u(x,t)t)}{1 + tg'(x - u(x,t)t)}$$

When the characteristics intersect, a **shock** is generated

The mathematical solution is **not unique**,
but the physical one is.

Rankine-Hugoniot conditions:

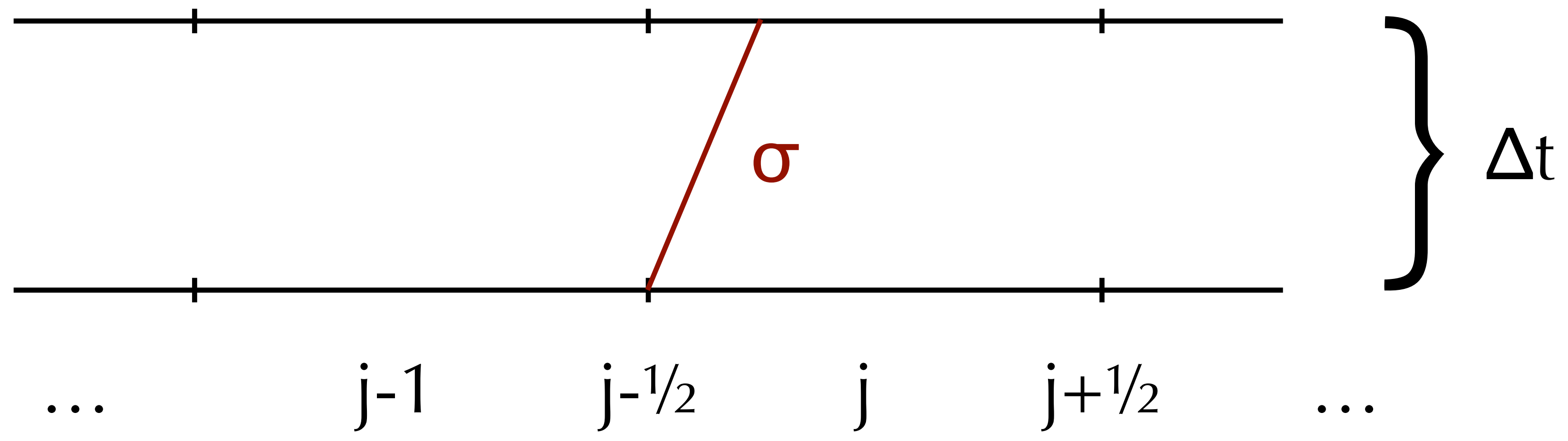
$$\begin{cases} \frac{dx}{dt} = \frac{f(u_R)-f(u_L)}{u_R-u_L} = \frac{u_R+u_L}{2} \\ x(t_s) = x_s \end{cases}$$

Roe's scheme for the state:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f(u_j^n) - f(u_{j-1}^n))$$

Scheme for the sensitivity:

$$u_{aj}^{n+1} = u_{aj}^n - \frac{\Delta t}{\Delta x}\frac{f(u_j^n) - f(u_{j-1}^n)}{u_j^n - u_{j-1}^n}(u_{aj}^n - u_{aj-1}^n)$$



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correction

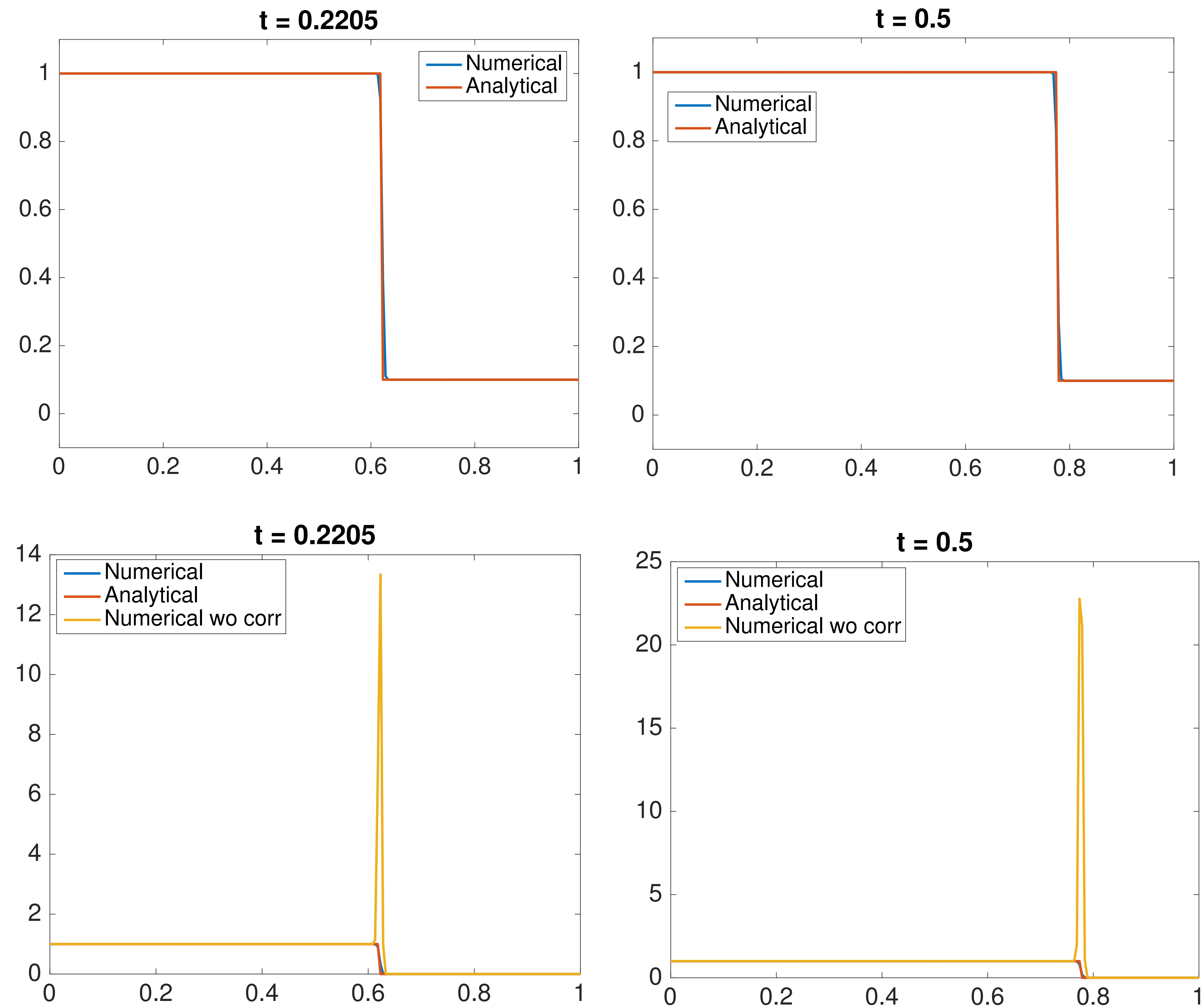
$$u_{aj}^{n+1} = u_{aj}^n - \frac{\Delta t}{\Delta x}(u_{aj}^n f'(u_j^n) - u_{aj-1}^n f'(u_{j-1}^n)) + \boxed{\frac{\Delta t}{\Delta x}\sigma_{aj}(u_j^n - u_{j-1}^n)}$$

$$\sigma_{aj}(u_j^n - u_{j-1}^n) = (u_{aj}^n f'(u_j^n) - u_{aj-1}^n f'(u_{j-1}^n)) - \frac{f(u_j^n) - f(u_{j-1}^n)}{u_j^n - u_{j-1}^n}(u_{aj}^n - u_{aj-1}^n).$$

The Riemann problem

$$g(x) = \begin{cases} u_L & x < x_c \\ u_R & x > x_c. \end{cases}$$

$$h(x) = \begin{cases} 1 & x < x_c \\ 0 & x > x_c. \end{cases}$$



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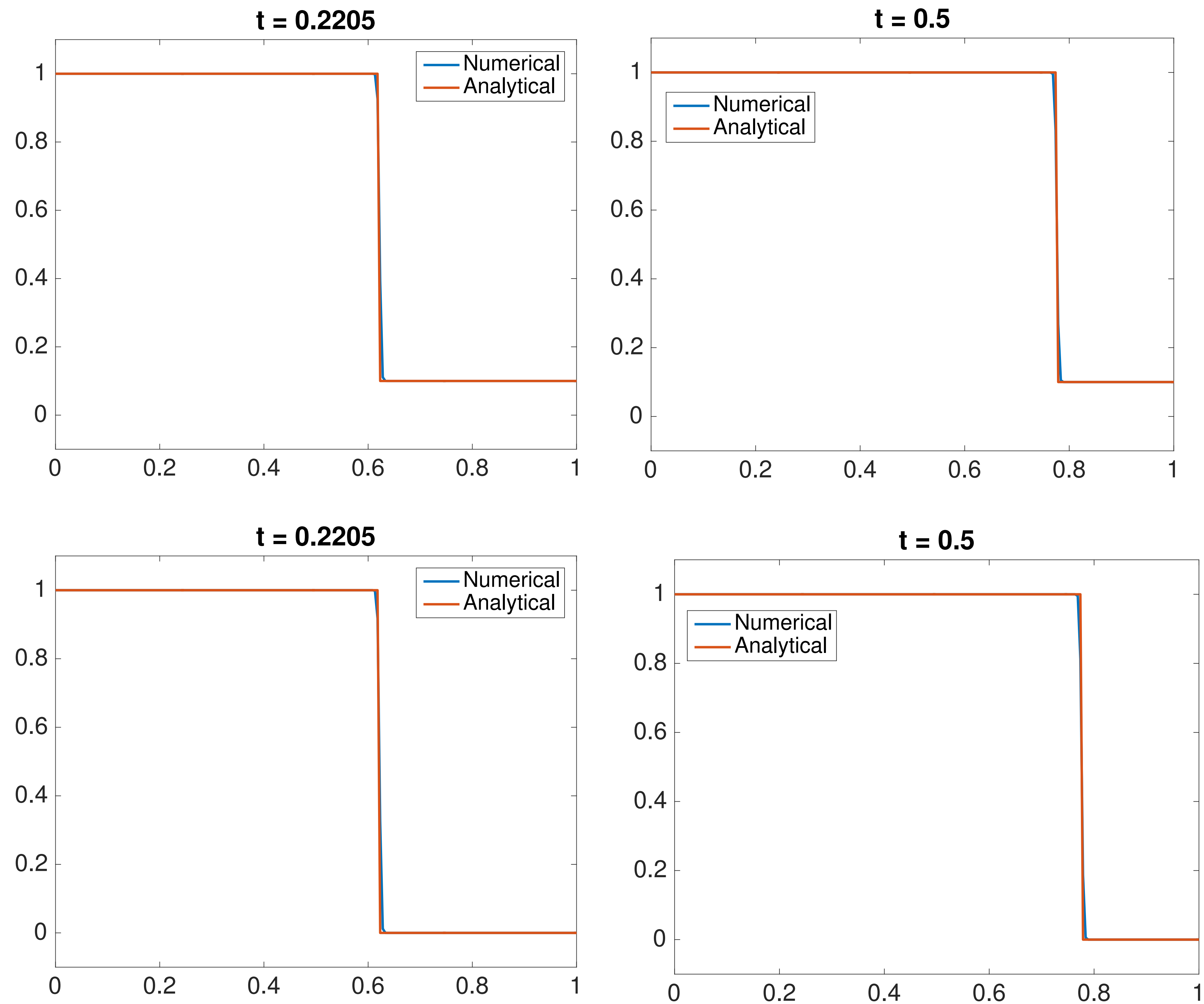
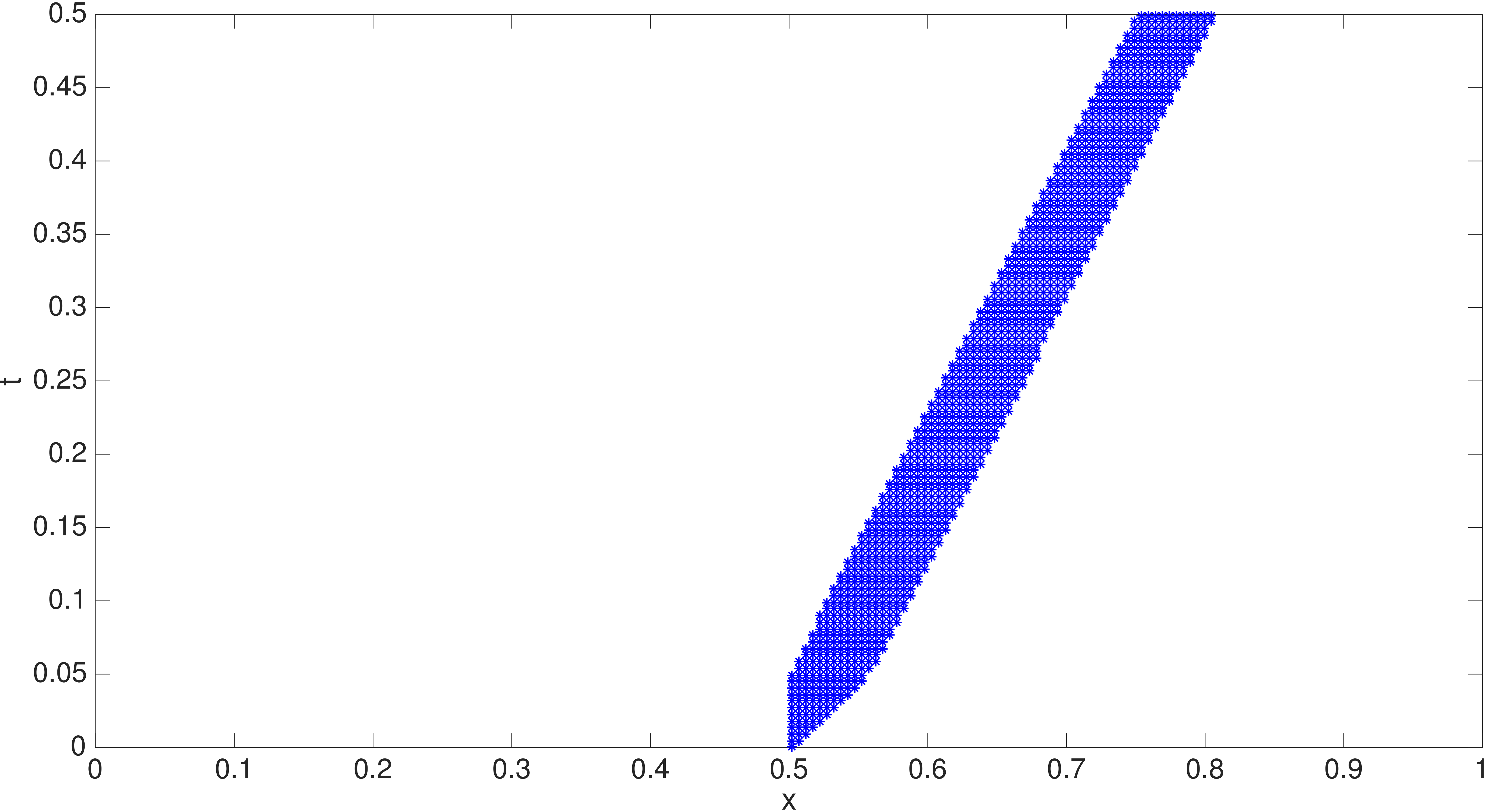
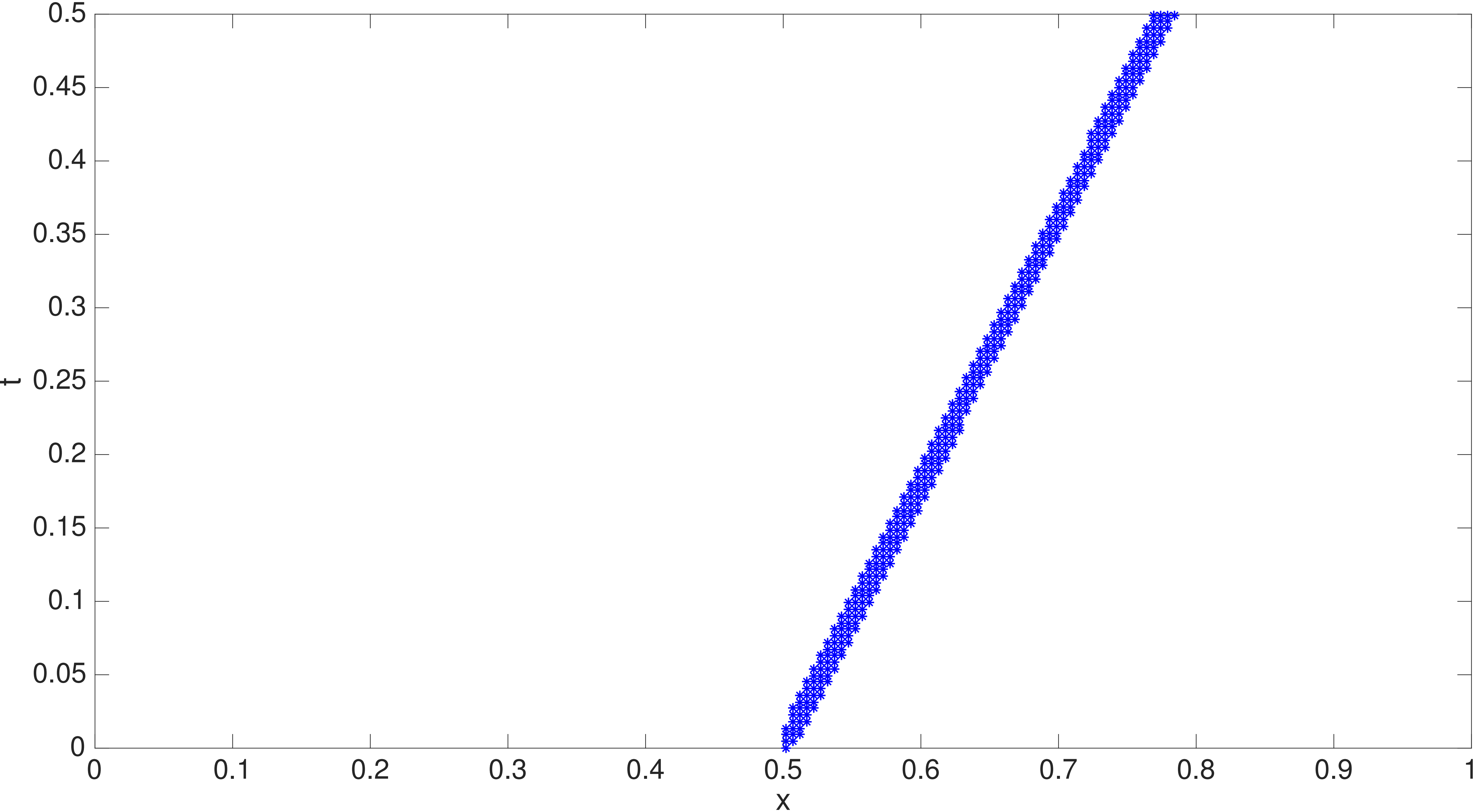


Numerical results



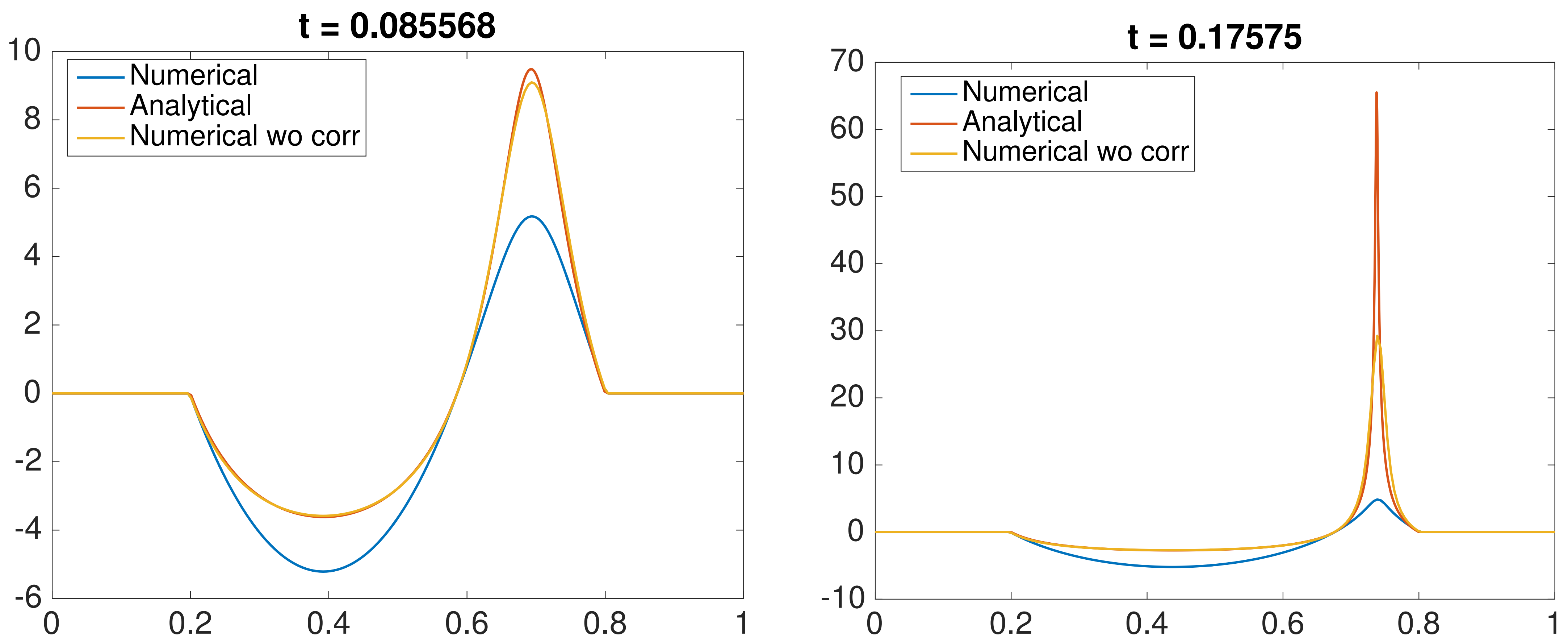
Numerical results

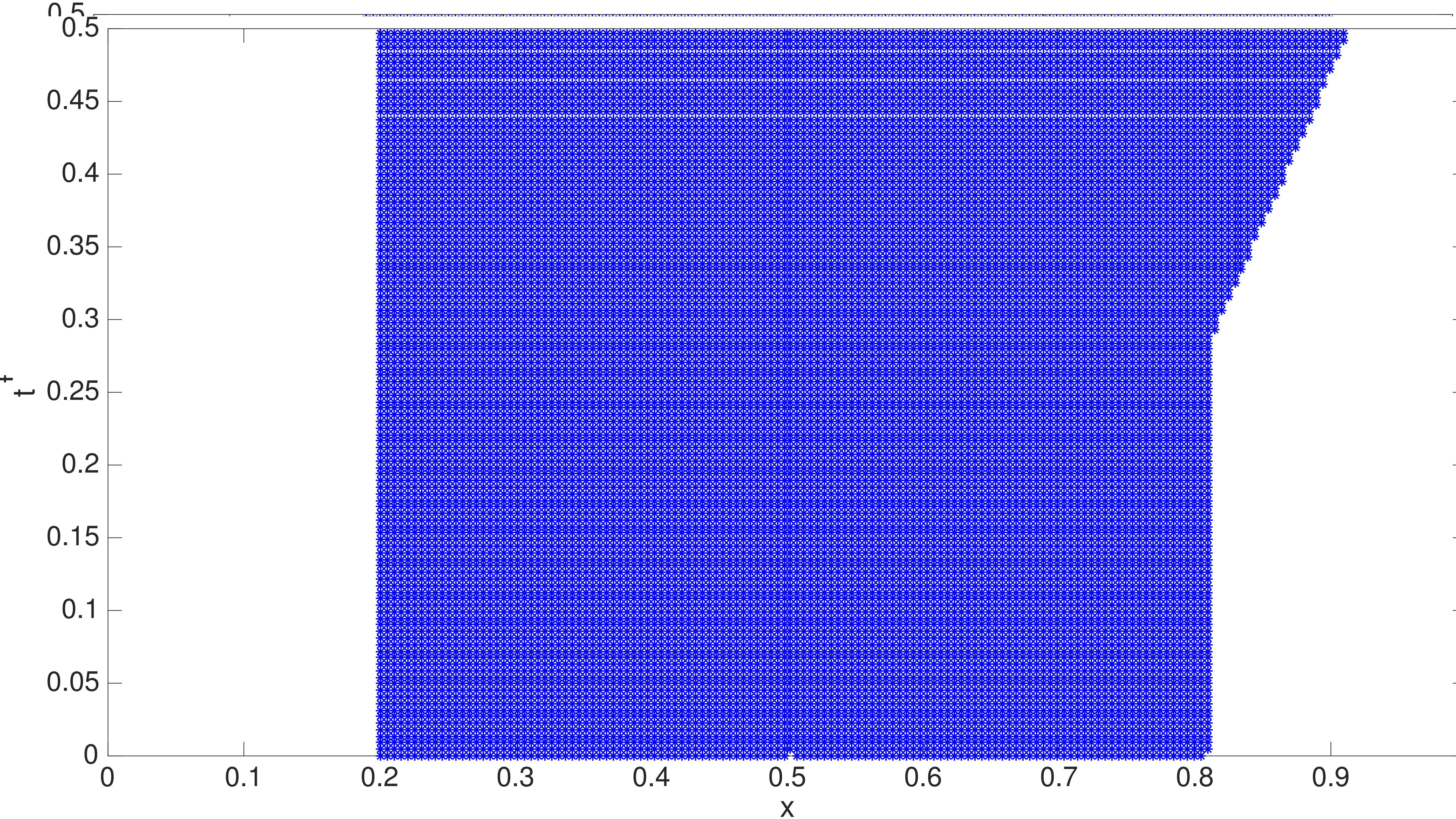


$$g(x) = \begin{cases} A\sin^2(\frac{\pi}{L}(x-x_c)+\frac{\pi}{2}) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{sinon.} \end{cases}$$

$$t_s \simeq 0.191 \qquad p = x_c$$

without shock detection

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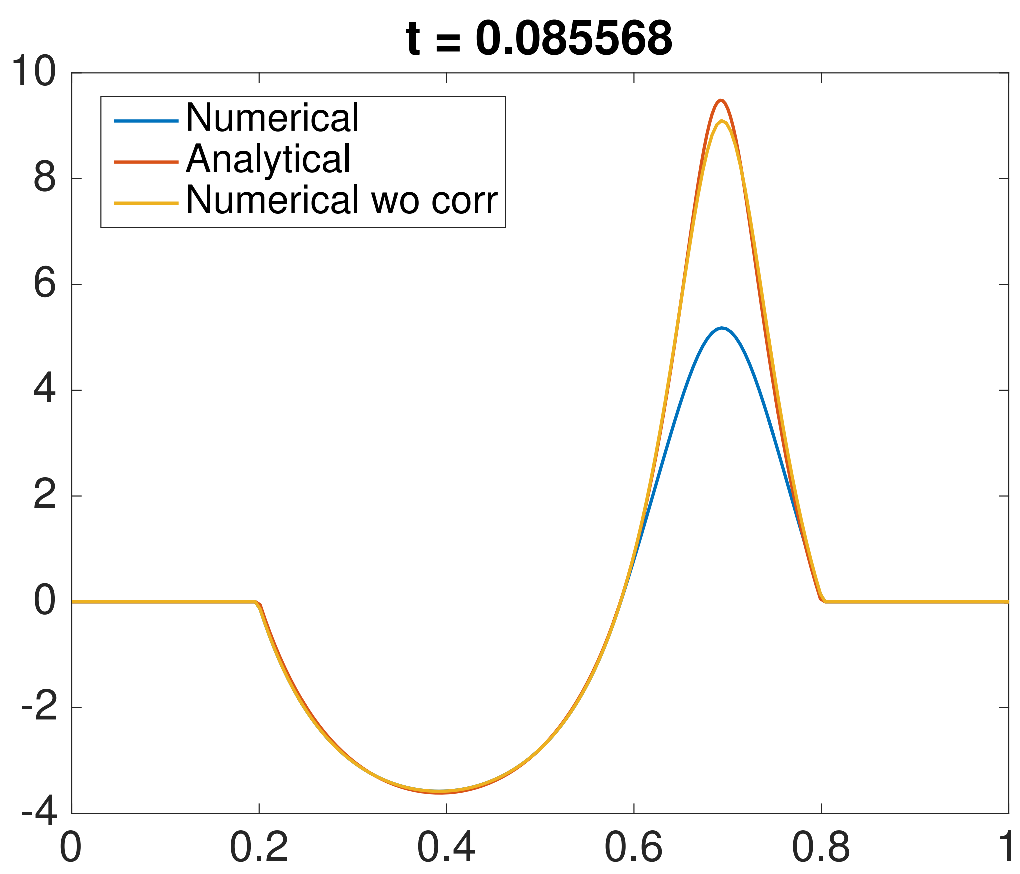
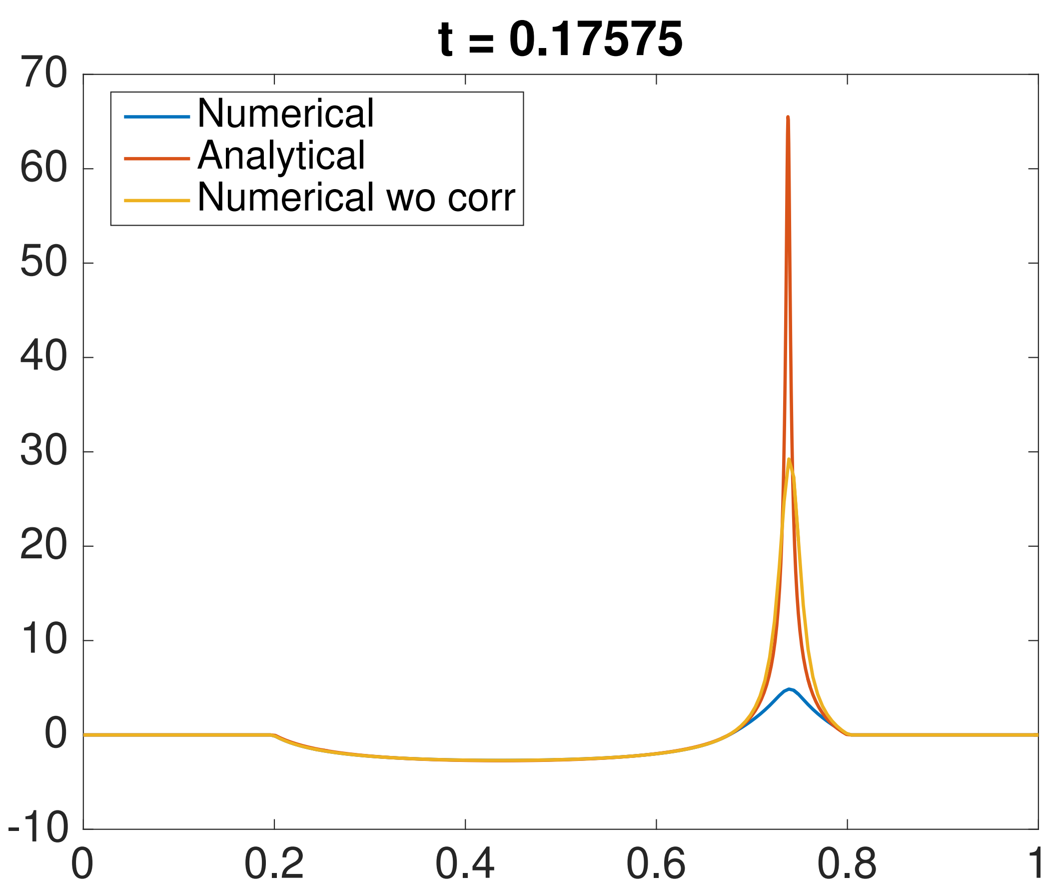
x

$$g(x) = \begin{cases} A\sin^2(\frac{\pi}{L}(x - x_c) + \frac{\pi}{2}) & x \in (x_c - \frac{L}{2}, x_c + \frac{L}{2}) \\ 0 & \text{sinon.} \end{cases}$$

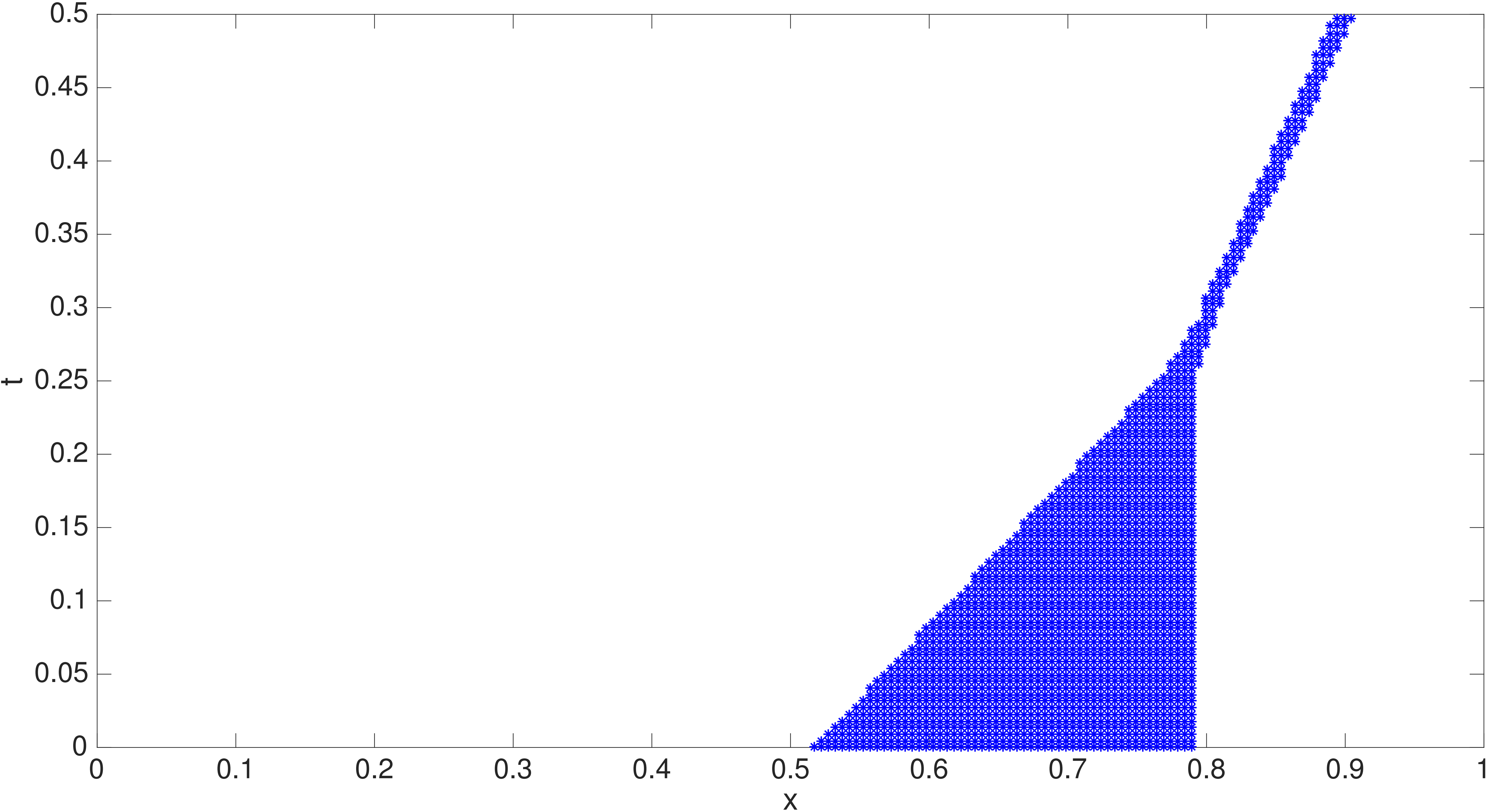
$$t_s \simeq 0.191 \qquad p = x_c$$

with a simple shock detector: $u_{j-1} > u_j > u_{j+1}$





Numerical results



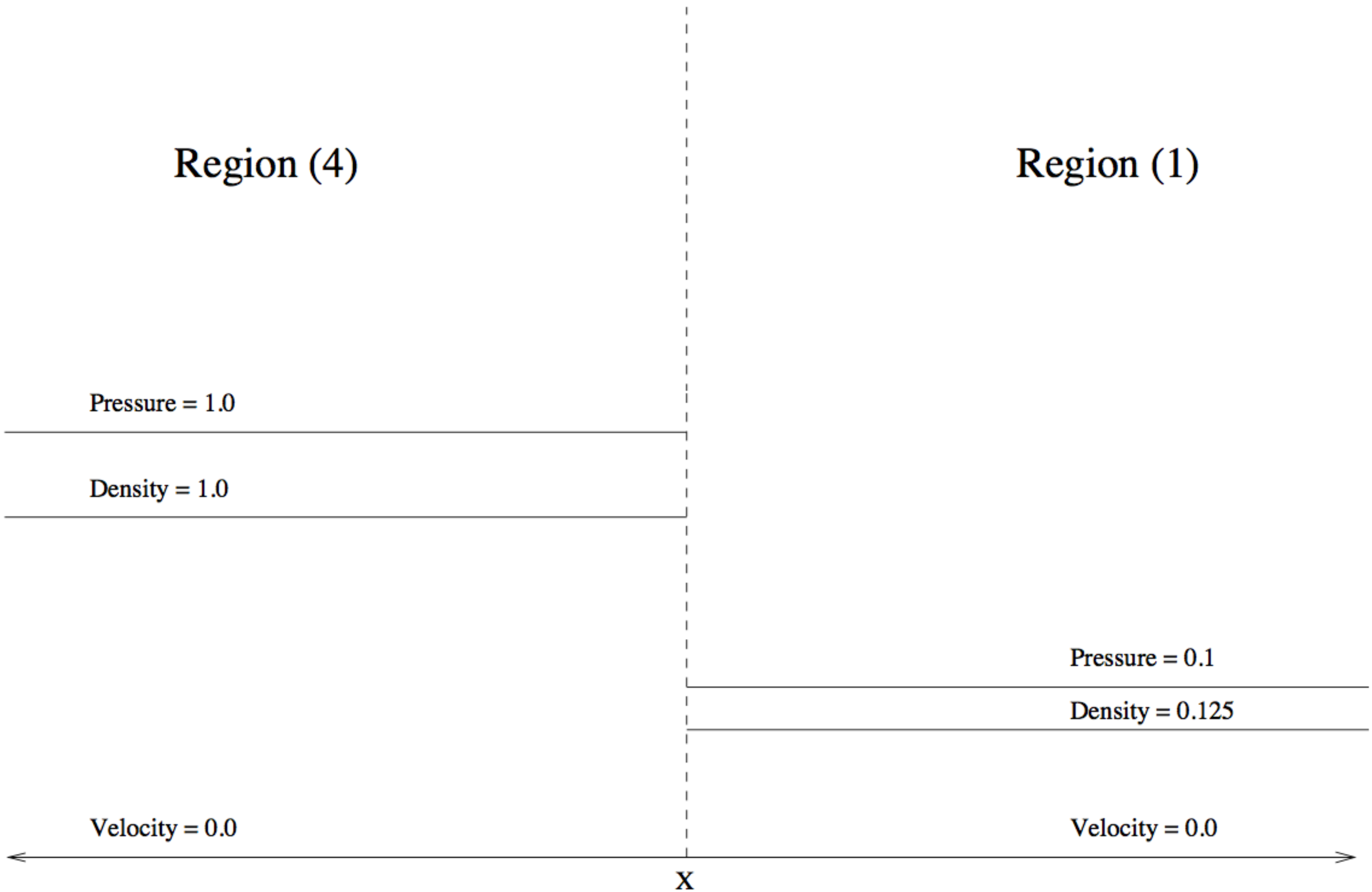
1D Euler system

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ E \\ s_\rho \\ s_{\rho u} \\ s_E \end{bmatrix} \qquad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \\ s_{\rho u} \\ s_\rho u^2 + 2\rho u s_u + s_p \\ s_u(E+p) + u(s_E + s_p) \end{bmatrix}$$

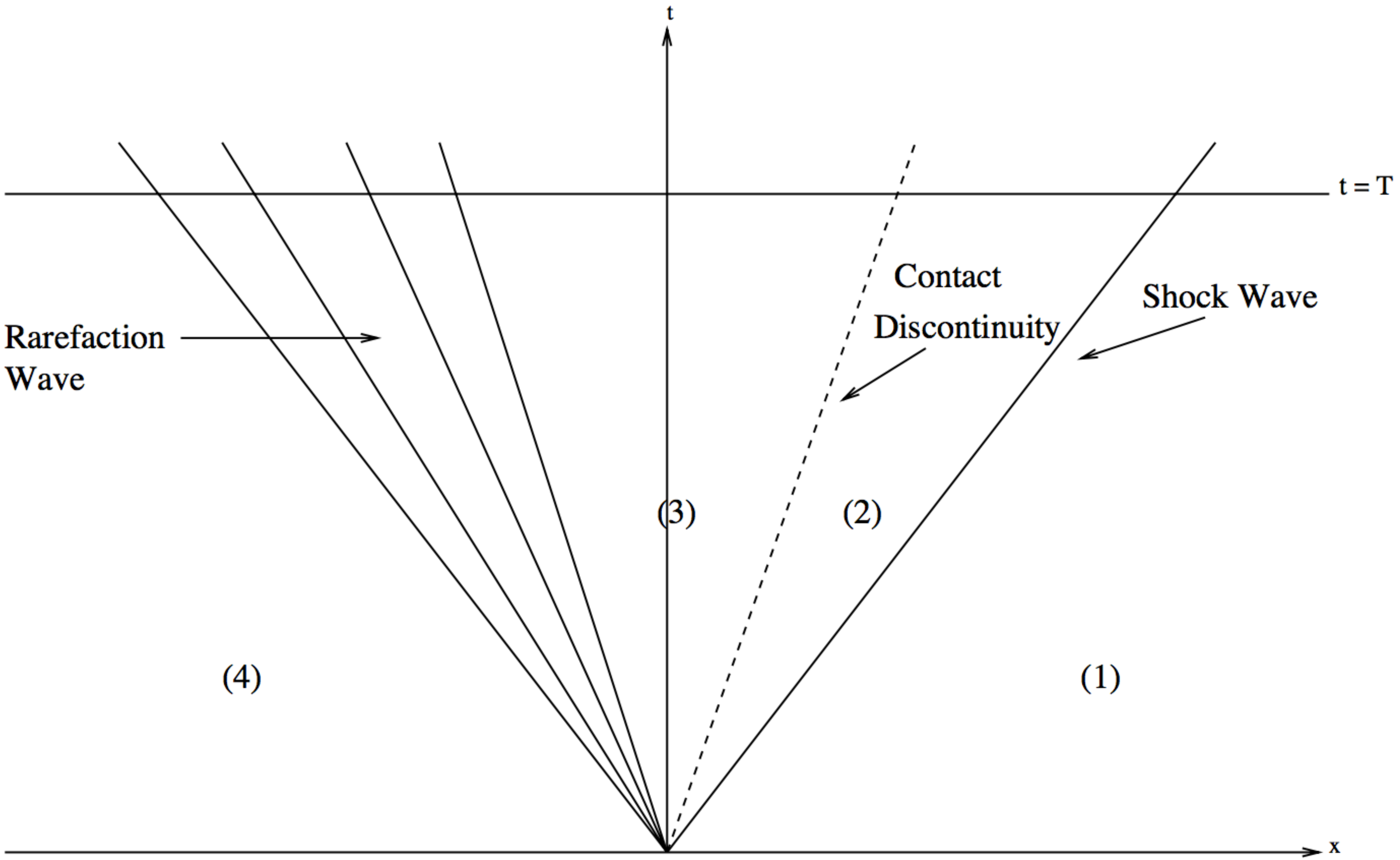
$$\frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} = \mathbf{M}(\mathbf{U}) = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$$

The matrix **M**(**U**) is not diagonalisable.

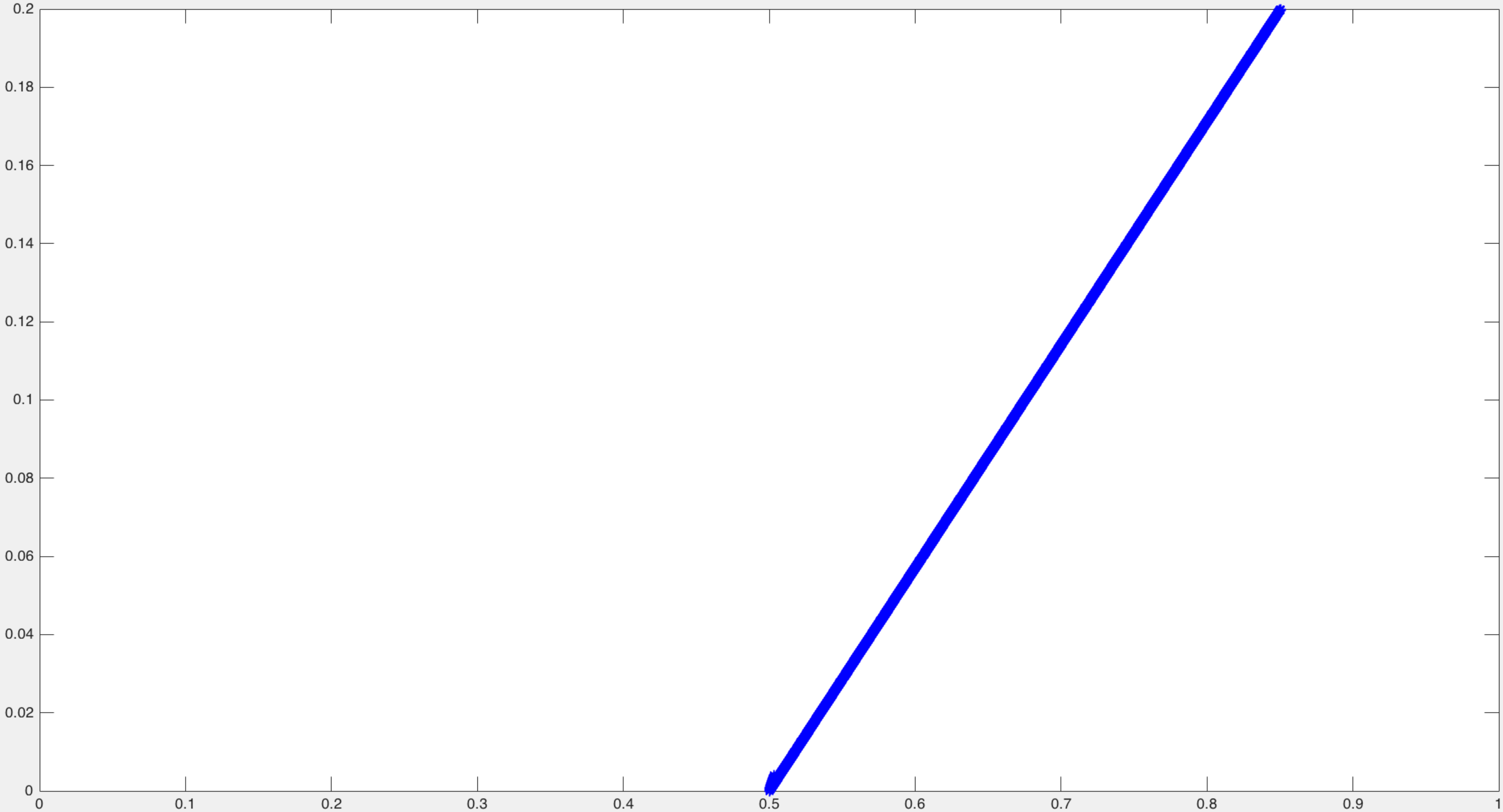
Riemann problem - analytical solution



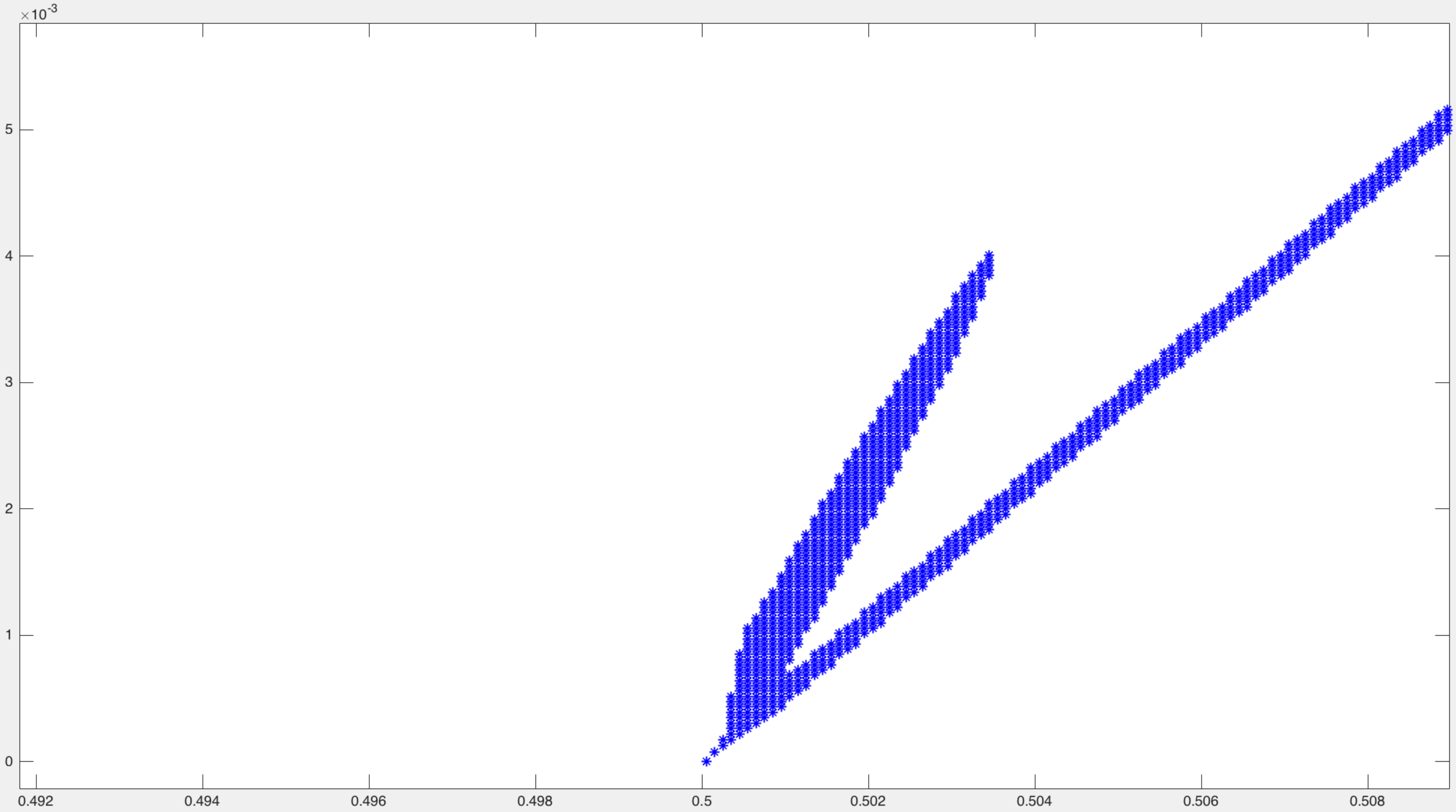
Riemann problem - analytical solution



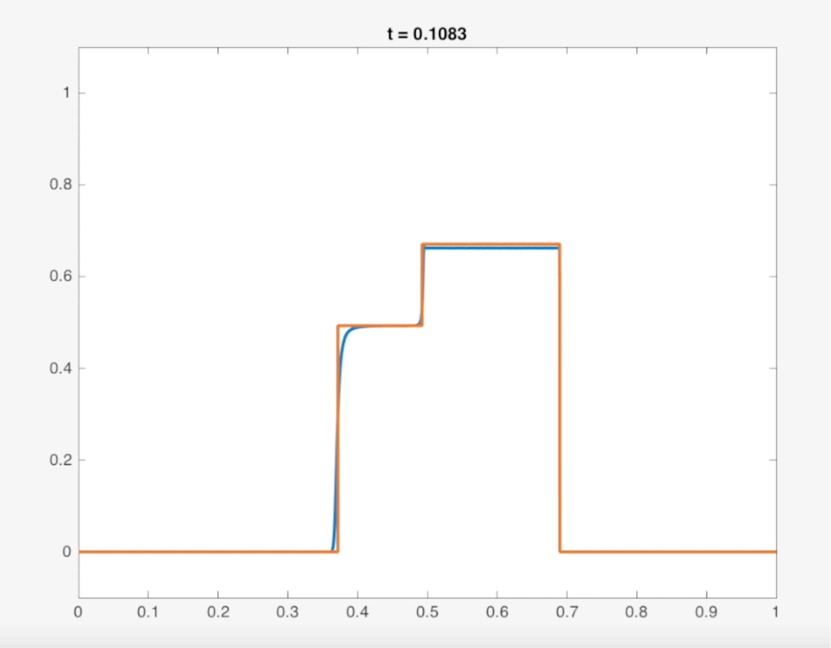
Riemann problem - shock detection

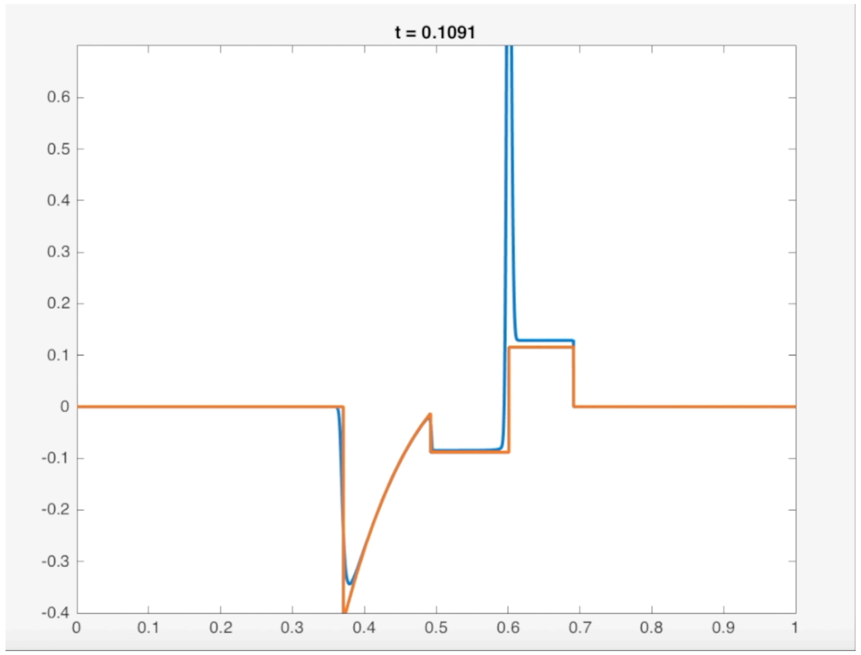


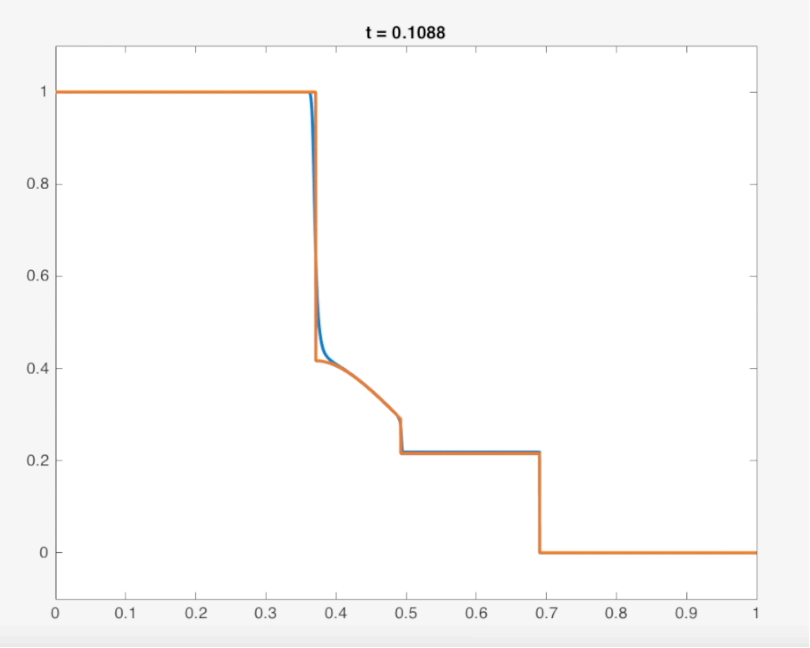
Riemann problem - shock detection



Shock tube







Velocity sensitivity | Density sensitivity | Pressure sensitivity

Future developments

- Develop better **shock detection** techniques
- Implement Roe (or a more accurate scheme) for **systems** and compare with other approximate Riemann solvers
- Sensitivity analysis for **Euler 2D/3D** equations

Thank you
for your attention